норме элемент во множестве $\partial^P \beta^0(p,r)$. *И наоборот, если при некотором достаточно большом* $c>0$ *существует ограниченная последовательность* $(\lambda^s,\mu^s)\in H\times R_+^m$, $s=1,2,\ldots$, *такая что для последовательности* $z^s\in\mathcal{D}$, $s=1,2,\ldots$, *элементы которой минимизируют с точностью* ε^s *модифицированную функцию Лагранжа* $L_{p,r}^{c,\delta^s}(z,\lambda^s,\mu^s)$, $z\in\mathcal{D}$ *с достаточно малыми* $\varepsilon^s>0$, *справедливы предельные соотношения* (*1*), *то выполняется и предельное соотношение* $f^0(z^s)\to\beta^0(p,r)$, $s\to\infty$. *При этом одновременно имеет место и предельное соотношение* $V_{p,r}^{c,0}(\lambda^s,\mu^s)\to\beta^0(p,r)$, $s\to\infty$.

Величина штрафного коэффициента $c>0$ определяется свойствами проксимального субградиента $\partial^P\beta^0(p,r)$. Отметим, что имеются примеры задач вида ($P_{p,r}^0$), для которых взятые формально без регуляризации элементы $(\lambda^s,\mu^s)=(\lambda_{p,r,c}^{\delta^s,0},\mu_{p,r,c}^{\delta^s,0})$, $s=1,2,\ldots$ не обеспечивают устойчивого построения МПР. Теорема 1 находит приложение при решении неустойчивых нелинейных задач оптимального управления.

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Kanatov A.V. ON STABLE SEQUENTIAL KUHN-TUCKER THEOREM IN NONLINEAR PROGRAMMING AND ITS APPLICATIONS
The stable with respect to the errors in the initial data sequential Kuhn–Tucker theorem in nondifferential form for parametric nonlinear mathematical programming problem in a Hilbert space and the possibility of its application for solving unstable optimal control problems are discussed.

Key words: nonlinear programming; sequential optimization; parametric problem; Kuhn–Tucker theorem in nondifferential form; dual regularization; optimal control; unstable problems.

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ON SOME EXTENSIONS OF OPTIMAL CONTROL THEORY

© **D.Yu. Karamzin, F.L. Pereira**

Key words: impulsive control; extension of classical theory.
Avoiding bringing sophisticated technique and without going into the rather complex details we describe the problem of extensions of the classical control theory on the introductory level.

It is a fact that classical calculus of variations problems might not have a smooth or even a continuous solution, although they are still of physical interest. Here, we shall focus namely on the discontinuous case. The following problem of variational calculus illustrates how discontinuities may emerge.

$$\begin{aligned} &\text{Minimize} && \int_0^1 x(t)\sqrt{1+(\dot{x})^2}dt, \\ &\text{subject to} && x(0)=R_1,\ x(1)=R_2. \end{aligned} \tag{15}$$

This is the problem of finding a minimal area of the surface of revolution formed by a membrane stretched over two parallel disks of radiuses R_1 and R_2, respectively. The application of the Euler-Lagrange principle leads to a second order differential equation and a boundary problem, which does not have a solution for some values of the parameters R_1, and R_2. The physical meaning of such situation is as follows: if numbers R_1, and R_2 are sufficiently large (or the distance between the two disks sufficiently small), the membrane exists and the surface of revolution is smooth. But, as the distance between the two disks increases, the membrane stretches and, at some point, breaks: at that very moment, the smooth and continuous solution fails to exist. However, this does not mean that the surface of revolution does not exist at all. Clearly, it is the surface of the union of the two disks aimed at each other and the segment $[0,1]$. This means that the solution $x(t)$ will be R_1 for $t=0$, R_2 for $t=1$ and 0 for $t\in(0,1)$ and, thus, it is discontinuous. In other words, the solution will be *impulsive*.

In the framework of his famous program David Hilbert suggested to extend calculus of variations theory in order to cover and to formulate such degenerate situations by giving a strict mathematical meaning to non-classical solutions. He expressed confidence that "every problem in the calculus of variations has a solution, provided that the term 'solution' is interpreted appropriately". This desideratum spawned a number of developments on the extension of the classic calculus of variation by various authors. For the rich history of this issue, we refer the reader to the article (B. Mordukhovich, Existence of optimal controls. J. Soviet Math. 7 (1977), 850-886). Here, we only point out to important contributions on extensions of the classical calculus of variations made by H. Lebesgue, L. Tonelli, L. Young, N.N. Bogolyubov, R.V. Gamkrelidze, V.F. Krotov, R.T. Rockafellar, V.M. Tikhomirov, J. Warga, among others.

With the advent of Optimal Control and the Pontryagin's maximum principle in the fifties, the theory of discontinuous solutions for variational calculus advanced significantly gradually giving shape to the area of Impulsive Optimal Control. So, what is the subject of the impulsive control theory? This theory covers and contains in itself, as a limiting case, a wide class of degenerate calculus of variations and optimal control problems for which classical continuous solutions fail to exist. This theory provides not only the way how to interpret the concept of solution but also the procedure to find it. The basic idea is to extend the conventional concept of control as well as the concept of trajectory. The usual bounded and measurable control can be replaced, for instance, by a Borel measure. Then, the trajectory becomes a function of bounded variation. This approach already suggests a reasonable extension for linear systems which covers many actual applications.

Let us give a simple example to illustrate how impulsive controls arise by providing an extension to the following calculus variation problem

$$\begin{aligned} &\text{Minimize} && \int_0^1 x^2 dt, \\ &\text{subject to} && \dot{x}=v,\ v\in\mathbb{R}^1, \\ & && x(0)=0,\ x(1)=1. \end{aligned} \tag{16}$$

So, we need to minimize the area under the curve $x^2(\cdot)$ where the arc $x(\cdot)$ is to reach the point $x=1$ starting from zero at $t=0$. Then, there is obviously no solution to this problem in the class of continuous trajectories due to the fact that any minimizing sequence of trajectories converges pointwise to the discontinuous function $x(t)=0$ as $t\in[0,1)$ and $x(1)=1$.

Then, how could a solution be defined? A solution can be found by extending the set of admissible trajectories admitting that the trajectory might have now discontinuities. Or, equivalently, this means to introduce impulsive controls instead of the convenient integrable control v from $\mathbb{L}_1$. For problem (16), it is appropriate to consider Borel measures as impulsive controls. Any conventional control $v(\cdot)$ can be considered as an absolutely continuous measure μ such that $d\mu=v(t)dt$. However, there will exist also other controls, like Dirac's measures, that cannot be reduced to conventional ones.

Thus, the conventional problem is extended by enlarging the class of trajectories/controls, being problem (16) rewritten in the impulsive context as follows:

$$\begin{aligned} &\text{Minimize} && \int_{[0,1]} x^2 dt, \\ &\text{subject to} && dx=d\mu, \quad \mu\in C^*([0,1]), \\ &&& x(0)=0,\ x(1)=1. \end{aligned}$$

Here, any admissible trajectory $x(\cdot)$ is already a function of bounded variation and so may exhibit discontinuities. The notation $dx=d\mu$ is understood in the integral sense, or, in terms of measures, it means that the Borel measure generated by $x(\cdot)$ is absolutely continuous with respect to μ and it is its Radon-Nykodim derivative with respect to μ (which, in this particular case, is equal to unity).

It is easy to see that solution to the extended problem exists and the optimal trajectory is $x(t)=0$ for $t<1$, and $x(1)=1$. Moreover, the fact that Borel measures are being used as extended controls, together with the weakly* sequential compactness of the unit ball in $C^*([0,1])$, implies that the described extension procedure is successful in ensuring the existence of solution for large classes of linear control problems. (For example, whenever the total variation $\int_0^1 |v(t)|dt$ is to be minimized.)

Thanks to the weak* convergence of measures, the extension procedure is rather clear when the dynamical system is linear in (x,v), like in the example (16). However, the complexity of the extension will increase when more general dynamical control systems are considered:

$$\dot{x} = f(x,u,t) + g(x,t)v, \quad v \in K, \tag{17}$$

where u is usual (classical) bounded control, the function f defines conventional control dynamics, v is an unbounded vector-valued control, g some matrix-valued function, and the set K, a convex closed cone.

How to describe the solution in this case? The extension procedure introduced above can not be applied any longer since the passage to the weak* limit is not correct for non-linear systems. Indeed, this is shown by the following simple example. Let $K=\mathbb{R}^2$. Consider the dynamical system with vector-valued control $v=(v_1,v_2)$:

$$\dot{x} = xv_1 + x^2 v_2, \quad x(0) = 1.$$

If we try to extend this system to the class of Borel measures, regarding bounded total variation $\int_0^1 |v(t)|dt \leqslant \text{const}$, we will see that, to every control, i.e., to every vector measure, there corresponds an entire integral funnel of trajectories $x(\cdot)$, any of which may claim to be called a solution to the extended dynamical system.

Thus, in the non-linear case, Borel measures are simply not enough to construct all achievable trajectories and controls. But, as we will see later, the new design control turns out be a Borel measure plus a certain family of usual measurable functions, which we will designate by *associated* family. The reason to introduce these associated functions is to select a single trajectory from the integral funnel and, thus, they can be regarded as controls acting at the discontinuities of the trajectory.

In the present report, we aim at extending the classic calculus of variations and/or optimal control problems by introducing a new type/design of impulsive controls. We will provide appropriate theorems for the existence of solution in constrained impulsive control problems.

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Карамзин Д.Ю., Перейра Ф.Л. О НЕКОТОРЫХ РАСШИРЕНИЯХ ТЕОРИИ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ

Без привлечения сложной техники и избегая достаточно трудоемких деталей мы описываем проблему расширений классической теории управления на вводном уровне.

Ключевые слова: импульсное управление; расширение классической теории.

УДК 517.977.1

РАВНОВЕСНЫЕ РЕШЕНИЯ В ОДНОЙ НЕАНТАГОНИСТИЧЕСКОЙ ПОЗИЦИОННОЙ ДИФФЕРЕНЦИАЛЬНОЙ ИГРЕ ДВУХ ЛИЦ

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Ключевые слова: неантагонистическая позиционная дифференциальная игра; векторные критерии; интегральные критерии.

В рассматриваемой игре с простой динамикой критерий качества первого игрока представляется суммой терминального и интегрального членов. Второй игрок имеет векторный критерий качества. Предполагается, что первый игрок действует в классе позиционных контрстратегий [1, 2], а второй – в классе чистых стратегий. Предложены понятия гарантированного выигрыша первого игрока и множества гарантированных выигрышей второго игрока [3, 4]. Дано определение равновесного решения игры нэшевского типа. Установлена структура таких решений.

Динамика игры описывается уравнением

$$\dot{x} = u + v, \quad x, u, v \in \mathbb{R}^2, \quad x(t_0) = x_0 \tag{1}$$

где x – фазовый вектор; управления u и v стеснены ограничениями $\|u\| \leqslant \mu$, $\|v\| \leqslant \mu$, $\mu > 0$; $[t_0, \vartheta]$ – заданный отрезок времени.

Функционалы качества игроков имеют вид:

$$I_1 = \|x(\vartheta)\| + \int\limits_{t_0}^{\vartheta} w(\tau)\|u(\tau) + v(\tau)\|^2 d\tau \to min, \tag{2}$$

$$I_2 = (-\|x(\vartheta) - a\|, \int\limits_{t_0}^{\vartheta} w(\tau)\|u(\tau) + v(\tau)\|^2 d\tau) \to max, \tag{3}$$